

THE SPECTRAL THEOREM FOR QUATERNIONIC UNBOUNDED NORMAL OPERATORS BASED ON THE S -SPECTRUM

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ABSTRACT. In this paper we prove the spectral theorem for quaternionic unbounded normal operators using the notion of S -spectrum. The proof technique consists of first establishing a spectral theorem for quaternionic bounded normal operators and then using a transformation which maps a quaternionic unbounded normal operator to a quaternionic bounded normal operator. With this paper we complete the foundation of spectral analysis of quaternionic operators. The S -spectrum has been introduced to define the quaternionic functional calculus but it turns out to be the correct object also for the spectral theorem for quaternionic normal operators. The fact that the correct notion of spectrum for quaternionic operators was not previously known has been one of the main obstructions to fully understanding the spectral theorem in this setting. A prime motivation for studying the spectral theorem for quaternionic unbounded normal operators is given by the subclass of unbounded anti-self adjoint quaternionic operators which play a crucial role in the quaternionic quantum mechanics.

1. INTRODUCTION

In the recent paper [6] a spectral theorem for quaternionic unitary operators based on the S -spectrum was proved using an extension of Herglotz's theorem to the quaternions. In this paper, inspired by [6], we treat the more general case of unbounded normal quaternionic operators.

The interest in spectral theory for quaternionic operators is motivated by the celebrated paper of Birkhoff and von Neumann, see [12], who showed that Schrödinger equation can be written only in the complex or quaternionic setting. Several authors, have given important contributions to the development of the quaternionic version of quantum mechanics, see [2, 19, 21, 26], but a correct notion of spectrum for quaternionic operators was still missing until the introduction of the S -spectrum, see, e.g., [16]. As it is well known, in the classical formulation of quantum mechanics the spectral theory of unbounded self-adjoint operators play a crucial role. In the fundamental paper [33], von Neumann used the spectral theorem for unitary operators to prove the spectral theorem for unbounded self-adjoint operators. In quaternionic quantum mechanics the most important quaternionic operators

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are unbounded anti self-adjoint operators; these operators are a particular case of quaternionic unbounded normal operators treated in this paper.

Our strategy to prove the spectral theorem is as follows: first we deduce the spectral theorem for quaternionic bounded normal operators. The proof is based on a continuous functional calculus defined in [22] and a classical version of the Riesz representation theorem. After we establish a spectral theorem for quaternionic bounded normal operators, we deduce a spectral theorem for quaternionic unbounded normal operators from the bounded case and from a suitable transformation.

With the quaternionic spectral theorem based on the S -spectrum we complete the foundation of the quaternionic spectral theory that started some years ago with the introduction of the S -functional calculus. In fact using the notion of slice hyperholomorphic functions, see [16], and the S -spectrum it is possible to define the quaternionic version of the Riesz-Dunford functional calculus which we now call quaternionic functional calculus or S -functional calculus.

We give a quick explanation of the reason why a consistent spectral theory for quaternionic operators is not so obvious. For simplicity consider a complex bounded operator $A : \mathcal{X} \rightarrow \mathcal{X}$ on a complex Banach space \mathcal{X} . The spectrum of A is defined as

$$\sigma(A) = \{\lambda \in \mathbb{C} : \lambda I_{\mathcal{X}} - A \text{ is not invertible in } B(\mathcal{X})\}$$

where $B(\mathcal{X})$ denotes the Banach space of all bounded linear operators on \mathcal{X} . Given a normal (bounded) linear operator T on a complex Hilbert space, in the spectral theorem

$$T = \int_{\sigma(T)} \lambda dE(\lambda)$$

the unique spectral measure $E(\lambda)$ associated to T is supported on $\sigma(T)$, see, e.g., [18]. The above notion of spectrum also appears in the Riesz-Dunford functional calculus, see [17], which is based on the Cauchy formula of holomorphic functions in which the Cauchy kernel is replaced by the resolvent operator $(\lambda I_{\mathcal{X}} - T)^{-1}$. Taking a holomorphic function h defined on an open set that contains the spectrum, we can use the Cauchy formula to define the linear operator $h(T)$.

From a historical view point a first attempt to generalize the classical notion of spectrum to quaternionic linear operators was to readapt the definition. To see the inconsistencies that occur consider a right linear quaternionic operator $T : \mathcal{V} \rightarrow \mathcal{V}$ acting on a quaternionic two-sided Banach space \mathcal{V} . The symbol $\mathcal{B}(\mathcal{V})$ denotes the Banach space of all bounded right linear quaternionic operators on \mathcal{V} . The left spectrum $\sigma_L(T)$ of T is related to the left-resolvent operator $(sI_{\mathcal{V}} - T)^{-1}$, i.e.,

$$\sigma_L(T) = \{s \in \mathbb{H} : sI_{\mathcal{V}} - T \text{ is not invertible in } \mathcal{B}(\mathcal{V})\},$$

where

$$(sI_{\mathcal{V}})(v) = sv, \quad v \in \mathcal{V}.$$

The right spectrum $\sigma_R(T)$ of T is associated with the right eigenvalue problem, i.e., the search for nonzero vectors v satisfying $T(v) = vs$. Observe that the operator $I_{\mathcal{V}}s - T$ associated with the right eigenvalue problem is not linear. Consequently, it is not clear what the resolvent operator ought to be. The quaternionic left-resolvent operator $(sI_{\mathcal{V}} - T)^{-1}$, as far as we know, is not hyperholomorphic in any sense. Consequently, the left-resolvent operator is not useful to define a hyperholomorphic quaternionic functional calculus. When we consider the right spectrum we just have

the notion of eigenvalues. The above discussion shows that there is a problem in adapting the classical notion of spectrum to either the left or right quaternionic spectrum.

As we shall see, relative to obtaining a spectral theorem for quaternionic normal operators, the appropriate notion of spectrum is a new notion of spectrum which is as follows. The S -spectrum, see [16], is defined as

$$\sigma_S(T) = \{s \in \mathbb{H} : T^2 - 2\operatorname{Re}(s)T + |s|^2 I_{\mathcal{V}} \text{ is not invertible in } \mathcal{B}(\mathcal{V})\},$$

where $s = s_0 + s_1 e_1 + s_2 e_2 + s_3 e_3$ is a quaternion, $\{1, e_1, e_2, e_3\}$ is the standard basis of the quaternions, $\operatorname{Re}(s) = s_0$ is the real part and the norm $|s| = \sqrt{s_0^2 + s_1^2 + s_2^2 + s_3^2}$.

We are now ready to illustrate our main result, the spectral theorem for normal quaternionic operators. We limit the discussion to the case of bounded normal operators but the theorem holds also for unbounded operators, see Theorem 6.2.

Consider the complex plane $\mathbb{C}_j := \mathbb{R} + j\mathbb{R}$, for $j \in \mathbb{S}$, where \mathbb{S} is the unit sphere of purely imaginary quaternions. Let \mathbb{C}_j^+ denote all $p \in \mathbb{C}_j$ with $\operatorname{Im}(p) \geq 0$. Observe that \mathbb{C}_j can be identified with a complex plane since $j^2 = -1$ for every $j \in \mathbb{S}$. If T be a (bounded) right linear normal operator on a quaternionic Hilbert space, then for $j \in \mathbb{S}$ there is a unique spectral measure E and a Hilbert basis \mathcal{N}_j of \mathcal{H} so that

$$T = \int_{\sigma_S(T) \cap \mathbb{C}_j^+} p dE(p).$$

To show the deep difference between complex spectral theory and the quaternionic spectral theory, we recall the quaternionic version of the Riesz-Dunford functional calculus, which suggests the notion of S -spectrum, see [14, 15]. This calculus involves two resolvent operators, namely a left and right S -resolvent operators given by

$$(1.1) \quad S_L^{-1}(s, T) := -(T^2 - 2\operatorname{Re}(s)T + |s|^2 I_{\mathcal{V}})^{-1}(T - \bar{s}I_{\mathcal{V}}), \quad s \in \rho_S(T)$$

and

$$(1.2) \quad S_R^{-1}(s, T) := -(T - \bar{s}I_{\mathcal{V}})(T^2 - 2\operatorname{Re}(s)T + |s|^2 I_{\mathcal{V}})^{-1}, \quad s \in \rho_S(T),$$

where $T \in \mathcal{B}(\mathcal{V})$ and $\rho_S(T) = \mathbb{H} \setminus \sigma_S(T)$ is the S -resolvent set. As one can see the S -spectrum is suggested by the S -resolvent operators.

Let $\Omega \subset \mathbb{H}$ be a suitable domain that contains the S -spectrum of T . We define the quaternionic functional calculus for left slice hyperholomorphic functions $f : \Omega \rightarrow \mathbb{H}$ as

$$(1.3) \quad f(T) = \frac{1}{2\pi} \int_{\partial(\Omega \cap \mathbb{C}_j)} S_L^{-1}(s, T) ds_j f(s),$$

where $ds_j = -dsj$; for right slice hyperholomorphic functions, we define

$$(1.4) \quad f(T) = \frac{1}{2\pi} \int_{\partial(\Omega \cap \mathbb{C}_j)} f(s) ds_j S_R^{-1}(s, T).$$

These definitions are well posed since the integrals depend neither on the open set Ω nor on the complex plane \mathbb{C}_j . Moreover, the resolvent equation, see [5], involves both S -resolvent operators, for s and $p \in \rho_S(T)$ we have

$$(1.5) \quad \begin{aligned} S_R^{-1}(s, T)S_L^{-1}(p, T) &= \{(S_R^{-1}(s, T) - S_L^{-1}(p, T))p - \bar{s}(S_R^{-1}(s, T) - S_L^{-1}(p, T))\} \\ &\quad \times (p^2 - 2s_0 p + |s|^2)^{-1}. \end{aligned}$$

Even though there are deep differences with respect to the classical resolvent equation for complex operators, all of the results that hold for the Riesz-Dunford functional calculus also hold for the quaternionic functional calculus. *We now claim that to replace the complex spectral theory with the quaternionic spectral theory we have to replace the classical spectrum with the S -spectrum.*

We conclude with some final remarks. In the case T is a right linear operator on a finite-dimensional Hilbert space, the S -spectrum of T coincides with the set of right eigenvalues of T ; in the general case of a linear operator, the point S -spectrum coincides with the set of right eigenvalues. In the literature the spectral theorem for quaternionic normal matrices based on the right spectrum is proved in [20]. In the literature, there are some papers on the quaternionic spectral theorem, see, e.g., [31, 21, 30, 32]. However, the notion of spectrum in the papers [21, 30, 32] is not made clear. In [23], a spectral theorem based on S -spectrum is proved for compact normal operators on a quaternionic Hilbert space. We point out that the S -resolvent operators are also used in Schur analysis in the realization of Schur functions in the slice hyperholomorphic setting see [1, 7, 8, 9] and [3, 10] for the classical case. In the papers [4, 13, 24] the problem of the generation of quaternionic groups and semigroups is treated using the S -spectrum.

The plan of the paper as follows: In Section 2 we give some preliminaries; in Section 3 we recall a continuous functional calculus for bounded normal operators; in Section 4 we prove the spectral theorem for bounded normal operators based on the S -spectrum; in Section 5 we introduce spectral integrals; finally, in Section 6 we prove the spectral theorem for unbounded normal operators based on the S -spectrum.

2. PRELIMINARIES

Let \mathcal{H} be a right linear quaternionic Hilbert space with an \mathbb{H} -valued inner product $\langle \cdot, \cdot \rangle$ which satisfies, for every $\alpha, \beta \in \mathbb{H}$, and $x, y, z \in \mathcal{H}$, the relations:

$$\begin{aligned} \langle x, y \rangle &= \overline{\langle y, x \rangle}. \\ \langle x, x \rangle &\geq 0 \quad \text{and} \quad \|x\|^2 := \langle x, x \rangle = 0 \iff x = 0. \\ \langle x\alpha + y\beta, z \rangle &= \langle x, z \rangle\alpha + \langle y, z \rangle\beta. \\ \langle x, y\alpha + z\beta \rangle &= \bar{\alpha}\langle x, y \rangle + \bar{\beta}\langle x, z \rangle. \end{aligned}$$

We call an operator $T : \mathcal{D}(\mathcal{H}) \rightarrow \mathcal{H}$ *right linear* if

$$T(x\alpha + y\beta) = (Tx)\alpha + (Ty)\beta,$$

for all x, y in the domain of T and $\alpha, \beta \in \mathbb{H}$. The set of right linear operators on \mathcal{H} will be denoted by $\mathcal{L}(\mathcal{H})$. Given $T \in \mathcal{L}(\mathcal{H})$, the domain of T will be denoted by $\mathcal{D}(T)$ and the range and kernel of T will be given by

$$\text{Ran}(T) = \{y \in \mathcal{H} : Tx = y \text{ for } x \in \mathcal{D}(\mathcal{H})\}$$

and

$$\text{Ker}(T) = \{x \in \mathcal{D}(T) : Tx = 0\},$$

respectively. We call an operator $T \in \mathcal{L}(\mathcal{H})$ *bounded* if

$$\|T\| := \sup_{\|x\| \leq 1} \|Tx\| < \infty.$$

In the sequel $\mathcal{B}(\mathcal{H})$ will denote the Banach space of all bounded right linear operators on \mathcal{H} endowed with the natural norm.

Definition 2.1. An operator $T \in \mathcal{L}(\mathcal{H})$ is called *closed* if the set $\{(x, Tx) : x \in \mathcal{H}\}$ is a closed subset of $\mathcal{H} \times \mathcal{H}$. Let S and T both belong $\mathcal{L}(\mathcal{H})$. We write $S = T$ if $\mathcal{D}(S) = \mathcal{D}(T)$ and $Sx = Tx$ for all $x \in \mathcal{D}(S) = \mathcal{D}(T)$. We write $S \subseteq T$ if $\mathcal{D}(S) \subseteq \mathcal{D}(T)$ and $Sx = Tx$ for all $x \in \mathcal{D}(S)$. Clearly, $S = T$ if and only if $S \subseteq T$ and $T \subseteq S$. An operator $T \in \mathcal{L}(\mathcal{H})$ is called *closable* if there exists a closed operator $\overline{T} := W \in \mathcal{L}(\mathcal{H})$ so that $T \subseteq W$.

Definition 2.2. Given $T \in \mathcal{L}(\mathcal{H})$ which is densely defined, we let $T^* \in \mathcal{L}(\mathcal{H})$ denote the unique operator so that

$$\langle Tx, y \rangle = \langle x, T^*y \rangle, \quad x \in \mathcal{D}(T).$$

The domain of T^* is given by

$$\mathcal{D}(T^*) = \{y \in \mathcal{H} : \text{there exists } z \in \mathcal{H} \text{ with } \langle Tx, y \rangle = \langle x, z \rangle\}.$$

Theorem 2.3. If $T \in \mathcal{L}(\mathcal{H})$ is densely defined and $W \in \mathcal{L}(\mathcal{H})$, then:

- (i) $T^* \in \mathcal{L}(\mathcal{H})$ is closed.
- (ii) $\text{Ran}(T)^\perp = \text{Ker}(T^*)$.
- (iii) If $T \subseteq W$, then $W^* \subseteq T^*$.

Proof. The proofs can be completed in much the same way as the case when \mathcal{H} is a complex Hilbert space (see, e.g., Proposition 1.6 in [29]). \square

Theorem 2.4. If $T \in \mathcal{L}(\mathcal{H})$ is densely defined, then:

- (i) T is closable if and only if $\mathcal{D}(T^*)$ is dense in \mathcal{H} .
- (ii) If T is closable, then $\overline{T} = T^{**}$.
- (iii) T is closed if and only if $\overline{T} = T^{**}$.
- (iv) If T is closable and $\text{Ker}(T) = \{0\}$, then T^{-1} is closable if and only if $\text{Ker}(\overline{T}) = \{0\}$. Moreover,

$$(\overline{T})^{-1} = \overline{T^{-1}}.$$

Proof. The proofs can be completed in much the same way as the case when \mathcal{H} is a complex Hilbert space (see, e.g., Theorem 1.8 in [29]). \square

Definition 2.5. Let $T \in \mathcal{L}(\mathcal{H})$. We call T *normal* if T is densely defined, T is closed and $TT^* = T^*T$.

Lemma 2.6. Let $T \in \mathcal{L}(\mathcal{H})$ be normal. If $S \in \mathcal{L}(\mathcal{H})$ so that $T \subseteq S$ and $\mathcal{D}(S) \subseteq \mathcal{D}(S^*)$, then $S = T$.

Proof. If $T \subseteq S$, then $S^* \subseteq T^*$ and hence

$$\mathcal{D}(T) \subseteq \mathcal{D}(S) \subseteq \mathcal{D}(S^*) \subseteq \mathcal{D}(T^*) = \mathcal{D}(T),$$

i.e., $\mathcal{D}(S) = \mathcal{D}(T)$. Therefore, $S = T$. \square

Definition 2.7. Let $T \in \mathcal{L}(\mathcal{H})$. We call T *self-adjoint*, *anti self-adjoint* and *unitary* if $T = T^*$, $T = -T^*$ and $TT^* = T^*T = I_{\mathcal{H}}$, respectively.

Definition 2.8. Let $T \in \mathcal{L}(\mathcal{H})$ be densely defined and let $\mathcal{R}_s(T) : \mathcal{D}(T^2) \rightarrow \mathcal{H}$ be given by

$$\mathcal{R}_s(T)x = \{T^2 - 2\operatorname{Re}(s)T + |s|^2 I_{\mathcal{H}}\}x, \quad x \in \mathcal{D}(T^2).$$

The S -resolvent set of T is defined as follows

$$\begin{aligned} \rho_S(T) &= \{s \in \mathbb{H} : \operatorname{Ker}(\mathcal{R}_s(T)) = \{0\}, \operatorname{Ran}(\mathcal{R}_s(T)) \text{ is dense in } \mathcal{H} \text{ and} \\ &\quad \mathcal{R}_s(T)^{-1} \in \mathcal{B}(\mathcal{H})\}. \end{aligned}$$

We recall some properties, see [16], of the S -spectrum. The S -spectrum satisfies

$$\sigma_S(T) = \mathbb{H} \setminus \rho_S(T).$$

Theorem 2.9. *Let $T \in \mathcal{B}(\mathcal{H})$. Then the S -spectrum is a compact non-empty subset of \mathbb{H} and*

$$(2.1) \quad \sigma_S(T) \subseteq \{p \in \mathbb{H} : 0 \leq |p| \leq \|T\|\}.$$

Proof. See Theorem 3.2.6 in [16]. \square

Theorem 2.10. *Let $T \in \mathcal{L}(\mathcal{H})$ be densely defined. If $p = p_0 + ip_1 \in \sigma_S(T)$ for $i \in \mathbb{S}$, then $p_0 + jp_1 \in \sigma_S(T)$ for all $j \in \mathbb{S}$.*

Proof. The proof of the assertion follows directly from the definition of the S -spectrum. If $s \in \sigma_S(T)$, then it follows immediately from the definition of $\sigma_S(T)$ that all the quaternions with the same real part and the same modulus belong to the S -spectrum of T . \square

Theorem 2.11. *Let $T \in \mathcal{L}(\mathcal{H})$. The following statements hold:*

- (i) *If T is positive, then $\sigma_S(T) \subseteq [0, \infty)$. If, in particular, $T \in \mathcal{B}(\mathcal{H})$ is positive, then*

$$\sigma_S(T) \subseteq [0, \|T\|].$$

- (ii) *If T is self-adjoint, then $\sigma_S(T) \subseteq \mathbb{R}$. If, in particular, $T \in \mathcal{B}(\mathcal{H})$ is self-adjoint, then*

$$\sigma_S(T) \subseteq [-\|T\|, \|T\|].$$

- (iii) *If T is anti self-adjoint, then $\sigma_S(T) \subseteq \{p \in \mathbb{H} : \operatorname{Re} p = 0\}$. If, in particular, $T \in \mathcal{B}(\mathcal{H})$ is anti self-adjoint, then*

$$\sigma_S(T) \subseteq \{p \in \mathbb{H} : \operatorname{Re} p = 0 \text{ and } |p| \leq \|T\|\}.$$

- (iv) *If T is unitary, then $\sigma_S(T) \subseteq \mathbb{S}$.*

Proof. If $T \in \mathcal{L}(\mathcal{H})$, then the containments illustrated in (i)-(iii) follow readily from the definition of $\sigma_S(T)$. If $T \in \mathcal{B}(\mathcal{H})$, then the containments illustrated in (i)-(iv) follow readily from (2.1). \square

We will also need the following version of the Riesz representation theorem.

Theorem 2.12. *Let X be a compact Hausdorff space and $\mathcal{C}(X, \mathbb{R})$ denote the normed space of real-valued continuous functions on X together with the supremum norm $\|\cdot\|_{\infty}$. Corresponding to any bounded positive linear functional $\psi : \mathcal{C}(X, \mathbb{R}) \rightarrow \mathbb{R}$ there exists a unique positive Borel measure μ on X such that*

$$(2.2) \quad \psi(f) = \int_X f(p) d\mu(p) \quad \text{for all } f \in \mathcal{C}(X, \mathbb{R}).$$

Proof. The assertion is a special case of Theorem D in Section 56 of [25]. \square

3. A FUNCTIONAL CALCULUS FOR BOUNDED NORMAL OPERATORS

Let \mathcal{H} be a right quaternionic Hilbert space and $\mathcal{B}(\mathcal{H})$ denote the set of all bounded right linear operators on \mathcal{H} . In the recent paper [22], Ghiloni, Moretti and Perotti established the existence of several functional calculi for a quaternionic bounded normal operator. Before introducing the functional calculus for bounded normal operators, we first need some notation and results.

Definition 3.1. Fix a Hilbert basis \mathcal{N} of a quaternionic Hilbert space \mathcal{H} . The *left scalar multiplication* L_p of \mathcal{H} induced by \mathcal{N} is the map

$$(p, x) \in \mathbb{H} \times \mathcal{H} \mapsto px \in \mathcal{H}$$

given by

$$px := \sum_{y \in \mathcal{N}} yp\langle x, y \rangle.$$

Lemma 3.2 (Statement (a) of Proposition 3.8 in [22]). *Let \mathcal{H} be a quaternionic Hilbert space. If $J \in \mathcal{B}(\mathcal{H})$ is an anti self-adjoint and unitary operator, then corresponding to any fixed $j \in \mathbb{S}$, there exists a left-scalar multiplication L_p so that*

$$J = L_j.$$

Theorem 3.3. *Let $T \in \mathcal{B}(\mathcal{H})$ be normal. Then there exist uniquely determined operators $A := (1/2)(T + T^*)$ and $B := (1/2)|T - T^*|$ which both belong to $\mathcal{B}(\mathcal{H})$ and an operator $J \in \mathcal{B}(\mathcal{H})$ which is uniquely determined on $\{\text{Ker}(T - T^*)\}^\perp$ so that the following properties hold:*

- (i) $T = A + JB$.
- (ii) A is self-adjoint and B is positive.
- (iii) J is anti self-adjoint and unitary.
- (iv) A , B and J mutually commute.
- (v) For any fixed $j \in \mathbb{S}$, there exists a Hilbert basis \mathcal{N}_j of \mathcal{H} with the property that $J = L_j$.

Proof. Properties (i)-(iv) appear in Theorem J on page 4 of [22]. Property (v) follows from Lemma 3.2. \square

Definition 3.4. Let $\Omega \subseteq \mathbb{H}$. We call Ω *axially symmetric* if for every point $p_0 + ip_1 \in \Omega$ with $i \in \mathbb{S}$, then $p_0 + jp_1 \in \Omega$ for all $j \in \mathbb{S}$.

Remark 3.5. Let $T \in \mathcal{L}(\mathcal{H})$. In view of Theorem 2.10, $\sigma_S(T)$ is an axially symmetric subset of \mathbb{H} .

Definition 3.6. Let $\mathcal{S}(\Omega, \mathbb{H})$ denote the quaternionic linear space of slice continuous functions on an axially symmetric subset Ω of \mathbb{H} , i.e., $\mathcal{S}(\Omega, \mathbb{H})$ consists of functions $f : \mathbb{H} \rightarrow \mathbb{H}$ of the form

$$f(u + vj) = \alpha(u, v) + j\beta(u, v), \quad j \in \mathbb{S},$$

where α and β are continuous \mathbb{H} -valued functions so that

$$\alpha(u, v) = \alpha(u, -v) \quad \text{and} \quad \beta(u, v) = -\beta(u, -v).$$

If α and β are real-valued, then we say that the continuous slice function f is *intrinsic*. The subspace of intrinsic continuous slice functions is denoted by $\mathcal{S}_{\mathbb{R}}(\Omega, \mathbb{H})$. The subspace of \mathbb{C}_j -valued functions in $\mathcal{S}_{\mathbb{R}}(\Omega, \mathbb{H})$ will be denoted by $\mathcal{S}_{\mathbb{R}}(\Omega, \mathbb{C}_j)$.

The following functional calculus will be useful for proving a spectral theorem for a normal operator $T \in \mathcal{B}(\mathcal{H})$.

Theorem 3.7 (Theorem 7.4 in [22]). *Let $T \in \mathcal{B}(\mathcal{H})$ be normal. There exists a unique continuous $*$ -homomorphism*

$$\Psi_{\mathbb{R},T} : f \in \mathcal{S}_{\mathbb{R}}(\sigma_S(T), \mathbb{H}) \mapsto f(T) \in \mathcal{B}(\mathcal{H})$$

of real-Banach unital C^* -algebras such that:

(i) $\Psi_{\mathbb{R},T}(\chi_{\sigma_S(T)}) = I_{\mathcal{H}}$, where

$$\chi_{\sigma_S(T)}(p) = \begin{cases} 1 & \text{if } p \in \sigma_S(T) \\ 0 & \text{if } p \notin \sigma_S(T). \end{cases}$$

(ii) $\Psi_{\mathbb{R},T}(\text{id}) = T$, where id denotes the inclusion map from $\sigma_S(T)$ to \mathbb{H} .

(iii) If J is as in Theorem 3.3, then J commutes with the normal operator $f(T)$.

(iv) If $f \in \mathcal{S}_{\mathbb{R}}(\sigma_S(T), \mathbb{H})$, then $\|f(T)\| = \|f\|_{\infty}$.

(v) If $f \in \mathcal{S}_{\mathbb{R}}(\sigma_S(T), \mathbb{H})$, then

$$(3.1) \quad \sigma_S(f(T)) = f(\sigma_S(T)).$$

Remark 3.8. For the convenience of the reader, we will now outline the construction of $f(T)$ for $f \in \mathcal{S}_{\mathbb{R}}(\sigma_S(T), \mathbb{H})$. Since $\sigma_S(T)$ is compact, there exist sequences of real-valued polynomials $\{\phi_n(u, v)\}_{n=0}^{\infty}$ and $\{\psi_n(u, v)\}_{n=0}^{\infty}$, so that

$$(3.2) \quad f_0(u, v) = \lim_{n \uparrow \infty} \phi_n(u, v) \text{ uniformly on } \sigma_S(T)$$

and

$$(3.3) \quad f_1(u, v) = \lim_{n \uparrow \infty} \psi_n(u, v) \text{ uniformly on } \sigma_S(T),$$

respectively.

The polynomials $\phi_n(u, v)$ and $\psi_n(u, v)$ can be constructed such that they are both slice continuous.

Since $\phi_n(u, v)$ and $\psi_n(u, v)$ have real coefficients and A and B are commuting self-adjoint operators it follows easily that $\phi_n(A, B) \in \mathcal{B}(\mathcal{H})$ and $\psi_n(A, B) \in \mathcal{B}(\mathcal{H})$ are self-adjoint. Next, we define

$$(3.4) \quad f_0(T)x := \lim_{n \uparrow \infty} \phi_n(A, B)x, \quad x \in \mathcal{H}$$

and

$$(3.5) \quad f_1(T)x := \lim_{n \uparrow \infty} \psi_n(A, B)x, \quad x \in \mathcal{H}.$$

Note that the limit in (3.4) exists since $\phi_n(A, B) = \phi_n(A, B)^*$ and hence

$$(3.6) \quad \begin{aligned} \|\{\phi_m(A, B) - \phi_n(A, B)\}x\|^2 &= \langle \{\phi_m(A, B) - \phi_n(A, B)\}^2 x, x \rangle \\ &\leq \|\{\phi_m(A, B) - \phi_n(A, B)\}^2\| \|x\|^2 \\ &= \|\phi_m - \phi_n\|_{\infty}^2 \|x\|^2 \\ &\rightarrow 0, \quad \text{as } m, n \uparrow \infty, \end{aligned}$$

since (3.2) holds. Note that item (iv) in Theorem 3.7 was used to obtain (3.6). The verification of the existence of the limit given in (3.5) is similar. The normal operator $f(T) \in \mathcal{B}(\mathcal{H})$ is given by

$$(3.7) \quad f(T) = f_0(T) + Jf_1(T).$$

Lemma 3.9. *Fix a normal operator $T \in \mathcal{B}(\mathcal{H})$. If $f = f_0 + f_1j \in \mathcal{S}_{\mathbb{R}}(\sigma_S(T), \mathbb{H})$, then $f_0(T)$ and $f_1(T)$ (given in (3.2) and (3.3), respectively) are self-adjoint.*

Proof. We claim that

$$(3.8) \quad \lim_{n \uparrow \infty} \langle \phi_n(A, B)x, y \rangle = \langle f_0(T)x, y \rangle, \quad x, y \in \mathcal{H},$$

and

$$(3.9) \quad \lim_{n \uparrow \infty} \langle \psi_n(A, B)x, y \rangle = \langle f_1(T)x, y \rangle, \quad x, y \in \mathcal{H}.$$

Assertion (3.8) follows directly from

$$\begin{aligned} |\langle \phi_n(A, B)x, y \rangle - \langle f_0(T)x, y \rangle| &\leq \|\phi_n(A, B) - f_0(T)\| \|x\| \|y\| \\ &= \|\phi_n - f_0\|_{\infty} \|x\| \|y\|, \end{aligned}$$

where item (iv) of Theorem 3.7 was used to obtain the last line. Assertion (3.9) is shown in much the same way. In view of (3.8),

$$\begin{aligned} \langle f_0(T)x, y \rangle &= \lim_{n \uparrow \infty} \langle \phi_n(A, B)x, y \rangle \\ &= \lim_{n \uparrow \infty} \langle x, \phi_n(A, B)y \rangle \\ &= \langle x, f_0(T)y \rangle, \quad x, y \in \mathcal{H}. \end{aligned}$$

Thus, $f_0(T)$ is self-adjoint. The fact that $f_1(T)$ is self-adjoint can be completed in much the same way using (3.9). \square

4. THE SPECTRAL THEOREM FOR BOUNDED NORMAL OPERATORS BASED ON THE S -SPECTRUM

In this section we shall consider normal operators T which are bounded, i.e., $T \in \mathcal{B}(\mathcal{H})$. We will generate a spectral theorem based on the S -spectrum using Theorems 2.12 and 3.7. This approach is analogous to a well-known approach in the classical case, i.e., when \mathcal{H} is a complex Hilbert space. See, e.g., the book of Lax [27] for details.

Fix a normal operator $T \in \mathcal{B}(\mathcal{H})$ and $j \in \mathbb{S}$. By Theorem 3.3, there exist commuting operators $A := (1/2)(T + T^*)$ and $B := (1/2)|T - T^*|$, where $|W| = (W^*W)^{1/2}$ for $W \in \mathcal{B}(\mathcal{H})$, and J which all belong to $\mathcal{B}(\mathcal{H})$ so that $T = A + JB$ and A and B are uniquely determined by T . Moreover, in view of item (v) of Theorem 3.3 we have the existence of a Hilbert basis \mathcal{N}_j of \mathcal{H} so that if J is written with respect to \mathcal{N}_j ,

$$(4.1) \quad J = L_j.$$

In what follows we shall assume, without loss of generality, that A , B , and J (and consequently T) are written with respect to \mathcal{N}_j .

Lemma 4.1. *Let $\mathcal{C}(\Omega_j^+, \mathbb{R})$ denote the set of real-valued continuous functions on $\Omega_j^+ = \sigma_S(T) \cap \mathbb{C}_j^+$ and $\mathcal{S}_{\mathbb{R}}(\Omega_j, \mathbb{R})$ denote the set of real-valued functions in $\mathcal{S}_{\mathbb{R}}(\Omega_j, \mathbb{H})$, where $\Omega_j = \sigma_S(T) \cap \mathbb{C}_j$. There exists a bijection between $\mathcal{C}(\Omega_j^+, \mathbb{R})$ and $\mathcal{S}_{\mathbb{R}}(\Omega_j, \mathbb{R})$. Moreover, there exists a bijection between $\mathcal{C}(\Omega_j^+, \mathbb{R})$ and purely imaginary functions in $\mathcal{S}_{\mathbb{R}}(\Omega_j, \mathbb{H})$.*

Proof. If $g \in \mathcal{C}(\Omega_j^+, \mathbb{R})$, then the function

$$\tilde{g}(u, v) = \begin{cases} g(u, v) & \text{if } u + jv \in \Omega_j^+ \\ g(u, -v) & \text{if } u + jv \in \Omega_j^- \end{cases}$$

belongs to $\mathcal{S}_{\mathbb{R}}(\Omega_j, \mathbb{R})$. Conversely, if $f \in \mathcal{S}_{\mathbb{R}}(\Omega_j, \mathbb{C}_j)$ is real-valued, then $\tilde{f} = f|_{\Omega_j^+} \in \mathcal{C}(\Omega_j^+, \mathbb{R})$.

The proof of the second assertion is completed in much the same way as the first assertion. \square

Fix $x \in \mathcal{H}$ and let

$$\ell_x(g) = \langle g(T)x, x \rangle, \quad g \in \mathcal{C}(\Omega_j^+, \mathbb{R}).$$

It is readily checked that ℓ_x is a real-valued bounded linear functional on the compact Hausdorff space $\mathcal{C}(\Omega_j^+, \mathbb{R})$. Moreover, ℓ_x is a positive functional. Indeed, if f is a continuous nonnegative function on Ω_j^+ , then g given by $g(u, v) = \sqrt{f(u, v)}$ also belongs to $\mathcal{C}(\Omega_j^+, \mathbb{R})$ and $g(T) = g(T)^*$. Thus,

$$\begin{aligned} \langle f(T)x, x \rangle &= \langle g(T)x, g(T)x \rangle \\ &= \|g(T)x\|^2 \geq 0. \end{aligned}$$

Theorem 2.12 yields the existence of a uniquely determined positive valued measure μ_x (for a fixed $j \in \mathbb{S}$) so that

$$(4.2) \quad \ell_x(g) = \int_{\Omega_j^+} g(p) d\mu_x(p), \quad g \in \mathcal{C}(\Omega_j^+, \mathbb{R}).$$

In view of (4.2), we may use the polarization formula

$$\begin{aligned} 4\langle Tx, y \rangle &= \langle T(x+y), x+y \rangle - \langle T(x-y), x-y \rangle + e_1 \langle T(x+ye_1), x+ye_1 \rangle \\ &\quad - e_1 \langle T(x-ye_1), x-ye_1 \rangle + e_1 \langle T(x-ye_2), x-ye_2 \rangle e_3 \\ &\quad - e_1 \langle T(x+ye_2), x+ye_2 \rangle e_3 + \langle T(x+ye_3), x+ye_3 \rangle e_3 \\ (4.3) \quad &- \langle T(x-ye_3), x-ye_3 \rangle e_3, \end{aligned}$$

where $\{1, e_1, e_2, e_3\}$ denotes the standard basis of \mathbb{H} , to obtain a uniquely determined quaternion-valued measure $\mu_{x,y}$ (relative to a fixed $j \in \mathbb{S}$) so that

$$(4.4) \quad \langle g(T)x, y \rangle = \int_{\Omega_j^+} g(p) d\mu_{x,y}(p), \quad g \in \mathcal{C}(\Omega_j^+, \mathbb{R}),$$

where

$$\begin{aligned} (4.5) \quad 4\mu_{x,y} &= \mu_{x+y} - \mu_{x-y} + e_1\mu_{x+ye_1} - e_1\mu_{x-ye_1} \\ &\quad + e_1\mu_{x-ye_2}e_3 - e_1\mu_{x+ye_2}e_3 + \mu_{x+ye_3}e_3 - \mu_{x-ye_3}e_3. \end{aligned}$$

Definition 4.2. The Borel sets of $\sigma_S(T) \cap \mathbb{C}_j^+$ will be denoted by $\mathfrak{B}(\sigma_S(T) \cap \mathbb{C}_j^+)$.

Lemma 4.3. The \mathbb{H} -valued measure $\mu_{x,y}$ given in (4.5) enjoys the following properties:

- (i) $\mu_{x\alpha+y\beta,z} = \mu_{x,z}\alpha + \mu_{y,z}\beta, \quad \alpha, \beta \in \mathbb{H}.$
- (ii) $\mu_{x,y\alpha+z\beta} = \bar{\alpha}\mu_{x,y} + \beta\mu_{x,z}, \quad \alpha, \beta \in \mathbb{H}.$
- (iii) $|\mu_{x,y}(\sigma_S(T) \cap \mathbb{C}_j^+)| \leq \|x\|\|y\|.$
- (iv) $\bar{\mu}_{x,y} = \mu_{y,x},$

for all $x, y, z \in \mathcal{H}$.

Proof. Properties (i)-(iii) are easily obtained from (4.4) using the uniqueness of $\mu_{x,y}$ (relative to a fixed $j \in \mathbb{S}$) and the properties of $\langle \cdot, \cdot \rangle$. Property (iv) follows from properties (i) and (ii). \square

It follows from Properties (i)-(iii) in Lemma 4.3 that $\Phi(x) = \mu_{x,y}(\sigma)$, where $y \in \mathcal{H}$ and $\sigma \in \mathfrak{B}(\sigma_S(T) \cap \mathbb{C}_j^+)$ are fixed, is a continuous right linear functional on \mathcal{H} . It follows from an analog of the Riesz representation theorem for Hilbert spaces (see Theorem 6.1 in [11]) that corresponding to any $x \in \mathcal{H}$, there exists a unique vector $w \in \mathcal{H}$ such that

$$\Phi(x) = \langle x, w \rangle,$$

i.e., $\mu_{x,y}(\sigma) = \langle x, w \rangle$. Using (i) and (ii) in Theorem 4.3, we get the existence of an operator $E \in \mathcal{B}(\mathcal{H})$ so that $w = E(\sigma)^*y$. Thus,

$$(4.6) \quad \mu_{x,y}(\sigma) = \langle E(\sigma)x, y \rangle, \quad \sigma \in \mathfrak{B}(\sigma_S(T) \cap \mathbb{C}_j^+).$$

In view of (4.4) and (4.6), we may write

$$(4.7) \quad g(T) = \int_{\sigma_S(T) \cap \mathbb{C}_j^+} g(p) dE(p), \quad g \in \mathcal{C}(\Omega_j^+, \mathbb{R}).$$

Theorem 4.4. *The $\mathcal{B}(\mathcal{H})$ -valued measure E , given by (4.7), enjoys the following properties:*

- (i) $\|E(\sigma)\| \leq 1$.
- (ii) $E(\emptyset) = 0$ and $E(\sigma_S(T) \cap \mathbb{C}_j^+) = I_{\mathcal{H}}$.
- (iii) If $\sigma \cap \tau = \emptyset$, then $E(\sigma \cup \tau) = E(\sigma) + E(\tau)$.
- (iv) $E(\sigma \cap \tau) = E(\sigma)E(\tau)$.
- (v) $E(\sigma)^* = E(\sigma)$.
- (vi) $E(\sigma)^2 = E(\sigma)$.
- (vii) $E(\sigma)$ commutes with $f(T)$ for all $f \in \mathcal{C}(\sigma_S(T) \cap \mathbb{C}_j^+, \mathbb{C}_j)$.
- (viii) $E(\sigma)$ and $E(\tau)$ commute for all $\sigma, \tau \in \mathfrak{B}(\sigma_S(T) \cap \mathbb{C}_j^+)$.

Proof. Property (i) follows directly from property (iii) in Lemma 4.3. Since $\mu_{x,y}(\emptyset) = 0$, we may use (4.6) to deduce $E(\emptyset) = 0$. Similarly, putting $g(p) = 1$ in (4.7) yields $g(T) = I_{\mathcal{H}}$ for all $x, y \in \mathcal{H}$ and thus

$$\langle x, y \rangle = \int_{\sigma_S(T) \cap \mathbb{C}_j^+} d\mu_{x,y} = \langle E(\sigma_S(T) \cap \mathbb{C}_j^+)x, y \rangle,$$

i.e., $E(\sigma_S(T) \cap \mathbb{C}_j^+) = I_{\mathcal{H}}$. Property (iv) follows easily from Property (i) of Lemma 4.3. Property (v) follows easily from property (iv) in Lemma 4.3. Property (vi) can be obtained from Property (iv) when $\sigma = \tau$. In view of the fact that $\sigma \cap \tau = \tau \cap \sigma$, Property (viii) can be obtained from Property (iv).

We will now show that Property (vii) holds. It follows from Theorem 3.3 that $T = A + JB$, where $A \in \mathcal{B}(\mathcal{H})$ is self-adjoint, $B \in \mathcal{B}(\mathcal{H})$ is positive and $J \in \mathcal{B}(\mathcal{H})$ is anti self-adjoint and unitary. Moreover, A , B and J all mutually commute. It follows from item (iii) in Theorem 3.7 that $f(T)$ commutes with J , and moreover, by the construction given in Remark 3.8, $f(T)$ commutes with A and B for $f \in \mathcal{S}_{\mathbb{R}}(\sigma_S(T), \mathbb{C}_j)$. In view of the identifications made in Lemma 4.1, we also have that $f(T)$ commutes with A and B for all $f \in \mathcal{C}(\sigma_S(T) \cap \mathbb{C}_j^+, \mathbb{C}_j)$, where $\mathcal{C}(\sigma_S(T) \cap$

$\mathbb{C}_j^+, \mathbb{C}_j$) denotes the set of \mathbb{C}_j -valued continuous functions on $\sigma_S(T) \cap \mathbb{C}_j^+$. To verify that $E(\sigma)$ commutes with A , note that

$$\langle f(T)Ax, y \rangle = \langle f(T)x, Ay \rangle \quad \text{for } x, y \in \mathcal{H}.$$

Thus, in view of (4.9) and (4.6), we have

$$\mu_{Ax, y} = \mu_{x, Ay}.$$

Consequently,

$$\langle E(\sigma)Ax, y \rangle = \langle E(\sigma)x, Ay \rangle = \langle AE(\sigma)x, y \rangle.$$

Thus, $E(\sigma)$ and A commute. In a similar fashion, one can show that $E(\sigma)$ and B commute and also that $E(\sigma)$ and J commute. Therefore, in view of (3.4), (3.5) and (3.7), we have that $E(\sigma)$ and $f(T)$ commute. \square

Definition 4.5. A $\mathcal{B}(\mathcal{H})$ -valued measure E on $\sigma_S(T) \cap \mathbb{C}_j^+$ will be called a *spectral measure* if E has Properties (i)-(viii) in Theorem 4.4. Note that $E(\sigma)$ is a positive operator for all $\sigma \in \mathfrak{B}(\sigma_S(T) \cap \mathbb{C}_j^+)$.

Remark 4.6. Fix $j \in \mathbb{S}$ and let \mathcal{N}_j be a Hilbert basis of \mathcal{H} so that when J is written with respect to \mathcal{N}_j , we have $J = L_j$ (such a basis exists by item (v) of Lemma 3.3). In Theorem 4.7 we shall assume that T is written with respect to the Hilbert basis \mathcal{N}_j .

In what follows we shall let $\mathcal{C}(\sigma_S(T) \cap \mathbb{C}_j^+, \mathbb{C}_j)$ denote the set of \mathbb{C}_j -valued continuous functions on $\sigma_S(T) \cap \mathbb{C}_j^+$. We are now ready to state and prove the main result of the section.

Theorem 4.7. *Let $T \in \mathcal{B}(\mathcal{H})$ be normal and fix $j \in \mathbb{S}$. If T is written with respect to the Hilbert basis \mathcal{N}_j (see Remark 4.6), then there exists a unique spectral measure E so that*

$$(4.8) \quad f(T) = \int_{\sigma_S(T) \cap \mathbb{C}_j^+} f(p) dE(p), \quad f \in \mathcal{C}(\sigma_S(T) \cap \mathbb{C}_j^+, \mathbb{C}_j).$$

Moreover, $W \in \mathcal{B}(\mathcal{H})$ commutes with A , B and J , which appear in the decomposition $T = A + BJ$ (see Theorem 3.3), if and only if W commutes with $E(\sigma)$, $\sigma \in \mathfrak{B}(\sigma_S(T) \cap \mathbb{C}_j^+)$.

Proof. If $f \in \mathcal{C}(\sigma_S(T) \cap \mathbb{C}_j^+, \mathbb{C}_j)$, then $f = f_0 + jf_1$, where f_0 and f_1 both belong to $\mathcal{C}(\sigma_S(T) \cap \mathbb{C}_j^+, \mathbb{R})$. In view of Lemma 4.1, f can be identified with a function in $\mathcal{S}_{\mathbb{R}}(\sigma_S(T), \mathbb{C}_j)$ (with a slight abuse of notation we will use f for the function in $\mathcal{S}_{\mathbb{R}}(\sigma_S(T), \mathbb{C}_j)$). Next, we have already observed in (3.7) that

$$f(T) = f_0(T) + Jf_1(T).$$

We will assume that $f(T)$ is written with respect to the Hilbert basis \mathcal{N}_j of \mathcal{H} so that $J = L_j$. Thus,

$$f(T) = f_0(T) + jf_1(T).$$

Thus, we can use (4.7) on $f_0(T)$ and $f_1(T)$ to obtain a spectral measure E on $\mathfrak{B}(\sigma_S(T) \cap \mathbb{C}_j^+)$ so that

$$f_0(T) = \int_{\sigma_S(T) \cap \mathbb{C}_j^+} f_0(p) dE(p),$$

$$f_1(T) = \int_{\sigma_S(T) \cap \mathbb{C}_j^+} f_1(p) dE(p)$$

and, finally,

$$(4.9) \quad f(T) = \int_{\sigma_S(T) \cap \mathbb{C}_j^+} f(p) dE(p), \quad f \in \mathcal{C}(\sigma_S(T) \cap \mathbb{C}_j^+, \mathbb{C}_j).$$

To see that the spectral measure E is unique for a fixed $j \in \mathbb{S}$, suppose not, i.e., there exists another spectral measure \tilde{E} on $\sigma_S(T) \cap \mathbb{C}_j^+$ with $\tilde{\mu}_{x,y}(\sigma) = \langle \tilde{E}(\sigma)x, y \rangle$, so that

$$f(T) = \int_{\sigma_S(T) \cap \mathbb{C}_j^+} f(p) dE(p) = \int_{\sigma_S(T) \cap \mathbb{C}_j^+} f(p) d\tilde{E}(p)$$

for $f \in \mathcal{C}(\sigma_S(T) \cap \mathbb{C}_j^+, \mathbb{C}_j)$. If we let

$$\begin{aligned} \Lambda(f) &= \langle f(T)x, x \rangle, \quad f \in \mathcal{C}(\sigma_S(T) \cap \mathbb{C}_j^+, \mathbb{C}_j), \\ &= \int_{\sigma_S(T) \cap \mathbb{C}_j^+} f(p) d\mu_{x,x}(p), \end{aligned}$$

then it is readily checked that Λ is a positive bounded \mathbb{C}_j -linear functional on $\mathcal{C}(\sigma_S(T) \cap \mathbb{C}_j^+, \mathbb{C}_j)$. It follows from the uniqueness assertion in the Riesz representation for complex-valued linear functionals, see, e.g., Theorem 6.19 in [28], that $\mu_{x,x} = \tilde{\mu}_{x,x}$. Using the polarization formula (4.3) we have

$$\mu_{x,y} = \tilde{\mu}_{x,y}, \quad x, y \in \mathcal{H},$$

and hence $E(\sigma) = \tilde{E}(\sigma)$ for $\sigma \in \mathfrak{B}(\sigma_S(T) \cap \mathbb{C}_j^+)$.

We will now prove the last assertion. If $W \in \mathcal{B}(\mathcal{H})$ commutes with A , B and J , then W commutes with $\psi(T)$ for any $\psi \in \mathcal{C}(\sigma_S(T) \cap \mathbb{C}_j^+, \mathbb{R})$. In view of (4.8),

$$\langle \psi(T)Wx, y \rangle = \int_{\sigma_S(T) \cap \mathbb{C}_j^+} \psi(p) d\langle E(p)Wx, y \rangle$$

and

$$\langle \psi(T)x, W^*y \rangle = \int_{\sigma_S(T) \cap \mathbb{C}_j^+} \psi(p) d\langle E(p)x, W^*y \rangle.$$

Thus, as

$$\langle \psi(T)Wx, y \rangle = \langle \psi(T)x, W^*y \rangle$$

we have

$$\langle E(\sigma)x, W^*y \rangle = \langle E(\sigma)Wx, y \rangle$$

and, consequently,

$$\langle WE(\sigma)x, y \rangle = \langle E(\sigma)Wx, y \rangle.$$

Therefore, $WE(\sigma) = E(\sigma)W$ for all $\sigma \in \mathfrak{B}(\sigma_S(T) \cap \mathbb{C}_j^+)$.

Conversely, suppose $W \in \mathcal{B}(\mathcal{H})$ and $E(\sigma)$ commute for all $\sigma \in \mathfrak{B}(\sigma_S(T) \cap \mathbb{C}_j^+)$. If $f(u + vj) = u$, $g(u + vj) = v$ and $h(u + vj) = j$, then f , g and h belong to $\mathcal{C}(\sigma_S(T) \cap \mathbb{C}_j^+, \mathbb{C}_j)$ and $f(T) = A$, $g(T) = B$ and $h(T) = J$. Consequently, using

(4.8), we have

$$\begin{aligned}
\langle AWx, y \rangle &= \int_{\sigma_S(T) \cap \mathbb{C}_j^+} u \, d\langle E(p)Wx, y \rangle \\
&= \int_{\sigma_S(T) \cap \mathbb{C}_j^+} u \, d\langle E(p)xW^*y \rangle \\
&= \langle Ax, W^*y \rangle \\
&= \langle WAx, y \rangle, \quad x, y \in \mathcal{H}.
\end{aligned}$$

Thus, W and A commute. The proof that W commutes with B and J is completed in a similar fashion. \square

Corollary 4.8. *In the setting of Theorem 4.7, the following statements hold:*

- (i) *If $T \in \mathcal{B}(\mathcal{H})$ is a positive operator, then there exists a unique positive operator $T^{1/2} := W \in \mathcal{B}(\mathcal{H})$ so that $W^2 = T$.*
- (ii) *$T \in \mathcal{B}(\mathcal{H})$ is self-adjoint if and only if*

$$(4.10) \quad T = \int_{[-\|T\|, \|T\|]} t \, dE(t).$$

- (iii) *$T \in \mathcal{B}(\mathcal{H})$ is anti self-adjoint if and only if*

$$(4.11) \quad T = \int_{[0, \|T\|]} jt \, dE(t).$$

- (iv) *$T \in \mathcal{B}(\mathcal{H})$ is unitary if and only if*

$$(4.12) \quad T = \int_{[0, \pi]} e^{jt} \, dE(t).$$

Proof. As we have observed in item (i) of Theorem 2.11, if $T \in \mathcal{B}(\mathcal{H})$ is a positive operator, then $\sigma_S(T) \subseteq [0, \|T\|]$. Thus, using Theorem 4.7 we have the existence of a uniquely determined spectral measure E so that

$$(4.13) \quad T = \int_{[0, \|T\|]} t \, dE(t).$$

Let $g(t) = t^{1/2}$ for $t \in \mathbb{R}$. Since $g \in \mathcal{C}(\sigma_S(T), \mathbb{R})$, it follows from Theorem 4.7 that

$$W := g(T) = \int_{[0, \|T\|]} t^{1/2} \, dE(t)$$

satisfies $W^2 = T$. Thus, we have established the existence of a positive operator $W \in \mathcal{B}(\mathcal{H})$ so that $W^2 = T$. The proof that W is unique follows from the uniqueness of the spectral measure E , just as in the case that \mathcal{H} is a complex Hilbert space.

The proofs of (ii)-(iv) follow readily from Theorem 4.7 and (2.1). \square

5. SPECTRAL INTEGRALS

The goal of this section is to extend the integral representation in Theorem 4.7 to a more general class of functions. This will be useful when proving a spectral theorem for unbounded operators in Section 6. To this end we will follow Chapter 4 of the book [29]. Most of the proofs of the properties of spectral integrals are easily adapted from the classical case presented in [29], i.e., when \mathcal{H} is a complex Hilbert space. However, some facts require additional arguments which we will highlight.

Fix a normal operator $T \in \mathcal{B}(\mathcal{H})$ and $j \in \mathbb{S}$. As in Section 4, we will assume throughout this section that T is written with respect to a Hilbert basis \mathcal{N}_j of \mathcal{H} so that $J = L_j$. The existence of such a Hilbert basis is guaranteed by item (v) of Theorem 3.3.

In view of Theorem 4.7 we have the existence of a uniquely determined spectral measure E so that (4.8) holds.

Definition 5.1. Let $\mathcal{B}(E, \Omega_j^+, \mathbb{C}_j)$, where $\Omega_j^+ = \sigma_S(T) \cap \mathbb{C}_j^+$, denote the set of all bounded E -measurable \mathbb{C}_j -valued functions f on Ω_j^+ with the norm

$$\|f\|_\infty = \sup_{p \in \Omega_j^+} |f(p)|.$$

Let $\mathcal{B}_s(E, \Omega_j^+, \mathbb{C}_j)$ denote the subset of simple functions in $\mathcal{B}(E, \Omega_j^+, \mathbb{C}_j)$, i.e., all \mathbb{C}_j -valued functions of the form $f(p) = \sum_{n=1}^k c_n \chi_{\sigma_n}(p)$, where $\sigma_1, \dots, \sigma_k$ are pairwise disjoint sets in $\mathfrak{B}(\Omega_j^+)$, $c_1, \dots, c_k \in \mathbb{C}_j$ and

$$\chi_\sigma(p) = \begin{cases} 1 & \text{if } p \in \sigma \\ 0 & \text{if } p \notin \sigma. \end{cases}$$

If $f \in \mathcal{B}_s(E, \Omega_j^+, \mathbb{C}_j)$, then we may define

$$(5.1) \quad f(T) = \sum_{n=1}^k c_n E(\sigma_n).$$

Lemma 5.2. If $f \in \mathcal{B}_s(E, \Omega_j^+, \mathbb{C}_j)$, then

$$(5.2) \quad \|f(T)\| \leq \|f\|_\infty.$$

Proof. If $f = \sum_{n=1}^k c_n \chi_{\sigma_n}$, where $c_1, \dots, c_k \in \mathbb{C}_j$ and $\sigma_1, \dots, \sigma_k$ are disjoint sets in $\mathfrak{B}(\Omega_j^+)$, then

$$\begin{aligned} \|f(T)x\|^2 &= \left\| \sum_{n=1}^k c_n E(\sigma_n)x \right\|^2 \\ &= \sum_{n=1}^k |c_n|^2 \|E(\sigma_n)x\|^2 \\ &\leq \|f\|_\infty^2 \|x\|^2. \end{aligned}$$

Thus, (5.2) holds. \square

Fix $f \in \mathcal{B}(E, \Omega_j^+, \mathbb{C}_j)$. Since $\mathcal{B}_s(E, \Omega_j^+, \mathbb{C}_j)$ is a dense subset of $\mathcal{B}(E, \Omega_j^+, \mathbb{C}_j)$, there exists a sequence of functions $\{f_n\}_{n=0}^\infty$ belonging to $\mathcal{B}_s(E, \Omega_j^+, \mathbb{C}_j)$ so that

$$\lim_{n \uparrow \infty} \|f_n - f\|_\infty = 0.$$

In view of (5.2), $\{f_n(T)x\}_{n=0}^\infty$ is a Cauchy sequence in \mathcal{H} . Let $f(T)$ be given by

$$f(T)x = \lim_{n \uparrow \infty} f_n(T)x, \quad x \in \mathcal{H}.$$

Note that f does not depend on the choice of the sequence $\{f_n\}_{n=0}^\infty$ and, consequently, neither does $f(T)$.

Lemma 5.3. If $f, g \in \mathcal{B}(E, \Omega_j^+, \mathbb{C}_j)$, where $\Omega_j^+ = \sigma_S(T) \cap \mathbb{C}_j^+$, $\alpha, \beta \in \mathbb{C}_j$ and $x, y \in \mathcal{H}$, then:

- (i) $f(T)^* = \bar{f}(T)$, $(\alpha f + \beta g)(T) = \alpha f(T) + \beta g(T)$.
- (ii) $\langle f(T)x, y \rangle = \int_{\Omega_j^+} f(t) d\langle E(p)x, y \rangle$.
- (iii) $\|f(T)x\|^2 = \int_{\Omega_j^+} |f(p)|^2 d\langle E(p)x, x \rangle$.
- (iv) $\|f(T)\| \leq \|f\|_\infty$.
- (v) $(fg)(T) = f(T)g(T)$.

Proof. In view of the density of $\mathcal{B}_s(E, \Omega_j^+, \mathbb{C}_j)$ in $\mathcal{B}(E, \Omega_j^+, \mathbb{C}_j)$ and (5.2), it suffices to check (i)-(v) when $f, g \in \mathcal{B}_s(E, \Omega_j^+, \mathbb{C}_j)$. Assertions (i)-(iv) are checked in a straightforward manner. If $f = \sum_{n=1}^k c_n \chi_{\sigma_n}$ and $g = \sum_{n=1}^k d_n \chi_{\tau_n}$ belong to $\mathcal{B}_s(E, \Omega_j^+, \mathbb{C}_j)$, then

$$\begin{aligned}
 (fg)(T) &= \sum_{m,n=1}^k c_m d_n E(\sigma_m \cap \tau_n) \\
 &= \sum_{m,n=1}^k c_m d_n E(\sigma_m) E(\tau_n) \\
 (5.3) \quad &= \left\{ \sum_{m=1}^k c_m E(\sigma_m) \right\} \left\{ \sum_{n=1}^k d_n E(\tau_n) \right\} \\
 &= f(T)g(T).
 \end{aligned}$$

Line (5.3) is justified by the fact that $J = L_j$ and, since J commutes with T , J must commute with $E(\sigma)$ for any $\sigma \in \mathfrak{B}(\Omega_j^+)$ (see the last assertion of Theorem 4.7). Consequently, $cE(\sigma) = E(\sigma)c$ for any $c \in \mathbb{C}_j$ and $\sigma \in \mathfrak{B}(\Omega_j^+)$. \square

We will now extend $\mathcal{B}(E, \Omega_j^+, \mathbb{C}_j)$ to a more general class which will be useful when proving the spectral theorem for unbounded normal operators.

Definition 5.4. Let $\mathcal{B}_\infty(E, \Omega_j^+, \mathbb{C}_j)$ denote the space of all E -measurable functions $f : \Omega_j^+ \rightarrow \mathbb{C}_j \cup \{\infty\}$ which satisfy

$$E(\{p \in \Omega_j^+ : f(p) = \infty\}) = 0.$$

Definition 5.5. A sequence of sets $\{\sigma_n\}_{n=0}^\infty$, where $\sigma_n \in \mathfrak{B}(\Omega_j^+)$ for $n = 0, 1, \dots$ is called a *bounding sequence* for a subset of functions $\mathfrak{F} \subseteq \mathcal{B}_\infty(E, \Omega_j^+, \mathbb{C}_j)$ if

- (i) For any $n = 0, 1, \dots$, f is bounded on σ_n .
- (ii) $\sigma_n \subseteq \sigma_{n+1}$ for $n = 0, 1, \dots$
- (iii) $E(\cup_{n=0}^\infty \sigma_n) = I_{\mathcal{H}}$.

Remark 5.6. If $\{\sigma_n\}_{n=0}^\infty$ is a bounded sequence, then the following assertions follow from Theorem 4.4:

- (i) $E(\sigma_n) \preceq E(\sigma_{n+1})$.
- (ii) $E(\sigma_n)x \rightarrow x$ as $n \uparrow \infty$, $x \in \mathcal{H}$.
- (iii) The set $\bigcup_{n=0}^\infty E(\sigma_n)\mathcal{H}$ is dense in \mathcal{H} .

We will now give meaning to $f(T)$ for $f \in \mathcal{B}_\infty(E, \Omega_j^+, \mathbb{C}_j)$.

Definition 5.7. Fix $f \in \mathcal{B}_\infty(E, \Omega_j^+, \mathbb{C}_j)$ and let $\{\sigma_n\}_{n=0}^\infty$ be a bounding sequence for f . We let

$$(5.4) \quad f(T)x = \lim_{n \uparrow \infty} (\chi_{\sigma_n} f)(T)x$$

with domain

$$(5.5) \quad \mathcal{D}(f(T)) = \{x \in \mathcal{H} : \int_{\Omega_j^+} |f(p)|^2 d\langle E(p)x, x \rangle < \infty\}.$$

Lemma 5.8. *If $f \in \mathcal{B}_\infty(E, \Omega_j^+, \mathbb{C}_j)$ and $\{\sigma_n\}_{n=0}^\infty$ is a bounding sequence for f , then:*

- (i) *A vector x belongs to $\mathcal{D}(f(T))$ if and only if the sequence $\{(\chi_{\sigma_n} f)(T)x\}_{n=0}^\infty$ converges in \mathcal{H} , or, equivalently,*

$$\sup_{n=0,1,\dots} \|(f\chi_{\sigma_n})(T)x\| < \infty.$$

- (ii) *$f(T)$ does not depend on the choice of bounding sequence for f .*
 (iii) *The set $\cup_{n=0}^\infty E(\sigma_n)\mathcal{H}$ is a dense subset of $\mathcal{D}(f(T))$. Moreover,*

$$(5.6) \quad E(\sigma_n)f(T) \subseteq f(T)E(\sigma_n) = (f\chi_{\sigma_n})(T), \quad n = 0, 1, \dots$$

Proof. We have already observed in Definition 4.5 that $E(\sigma)$ is a positive operator on Ω_j^+ for every $\sigma \in \mathfrak{B}(\Omega_j^+)$. Thus, μ_x is a positive measure on Ω_j^+ , where $\mu_x(\sigma) = \langle E(\sigma)x, x \rangle$. Consequently, the proof of items (i)-(iii) can be completed in much the same way as in items (i)-(iii) of Theorem 4.13 in [29]. \square

In the following theorem, \overline{W} shall denote the closure of an operator $W \in \mathcal{L}(\mathcal{H})$.

Theorem 5.9. *If $f, g \in \mathcal{B}_\infty(E, \Omega_j^+, \mathbb{C}_j)$ and $\alpha, \beta \in \mathbb{C}_j$, then:*

- (i) $\bar{f}(T) = f(T)^*$.
 (ii) $(\alpha f + \beta g)(T) = \overline{\alpha f(T) + \beta g(T)}$.
 (iii) $(fg)(T) = \overline{f(T)g(T)}$.
 (iv) $f(T)$ is a closed normal operator on \mathcal{H} and

$$f(T)^* f(T) = (f\bar{f})(T) = (\bar{f}f)(T).$$

- (v) $\mathcal{D}(f(T)g(T)) = \mathcal{D}(g(T)) \cap \mathcal{D}((fg)(T))$.

Proof. The proof of items (i)-(iv) when \mathcal{H} is a complex Hilbert space (see items (i)-(v) of Theorem 4.16 in [29]) can easily be adapted to the case when \mathcal{H} is a quaternionic Hilbert space. \square

Theorem 5.10. *If $f \in \mathcal{B}_\infty(E, \Omega_j^+, \mathbb{C}_j)$, then $f(T)$ is invertible if and only if f does not vanish E -a.e. on Ω_j^+ . In this case,*

$$(5.7) \quad f(T)^{-1} = (1/f)(T),$$

where we use the convention that $1/0 = \infty$ and $1/\infty = 0$.

Proof. The proof when \mathcal{H} is a complex Hilbert space (see Proposition 4.19 in [29]) can easily be adapted to the case when \mathcal{H} is a quaternionic Hilbert space. \square

Lemma 5.11. *If $g : \Omega_j^+ \rightarrow \tilde{\Omega}_j^+ \subseteq \mathbb{C}_j$, $h \in \mathcal{B}_\infty(\tilde{E}, \tilde{\Omega}_j^+, \mathbb{C}_j)$ and \tilde{E} is the spectral measure on $\tilde{\Omega}_j^+$ given by*

$$\tilde{E}(\sigma) = E(g^{-1}(\sigma)), \quad \sigma \in \mathfrak{B}(\tilde{\Omega}_j^+),$$

then $h \circ g \in \mathcal{B}_\infty(E, \Omega_j^+, \mathbb{C}_j)$ and

$$(5.8) \quad \int_{\tilde{\Omega}_j} h(p) d\tilde{E}(p) = \int_{\Omega_j^+} h(g(p)) dE(p).$$

Proof. The proof is similar to the proof when \mathcal{H} is a complex Hilbert space (see Proposition 4.24 in [29]) except for the fact that the polarization formula (4.3) needs to be used. \square

6. THE SPECTRAL THEOREM FOR UNBOUNDED NORMAL OPERATORS BASED ON THE S -SPECTRUM

In this section we will consider normal operators T which are unbounded, i.e., $T \in \mathcal{L}(\mathcal{H})$ but $T \notin \mathcal{B}(\mathcal{H})$. The strategy will be to transform T into a normal operator $Z_T \in \mathcal{B}(\mathcal{H})$ and use Theorem 4.7 and a change of variable argument to obtain a spectral theorem for T based on the S -spectrum. Obtaining a spectral theorem for unbounded operators in the aforementioned way has been done in the classical case, i.e., when \mathcal{H} is a complex Hilbert space. See, e.g., the book of Schmüdgen [29].

Given $T \in \mathcal{L}(\mathcal{H})$, we let

$$(6.1) \quad Z_T = TC_T^{1/2},$$

where $C_T = (I_{\mathcal{H}} + T^*T)^{-1}$. Note that C_T is a bounded positive operator on \mathcal{H} and

$$(6.2) \quad C_T = I_{\mathcal{H}} - Z_T^*Z_T.$$

Formula (6.2) follows from

$$\begin{aligned} I_{\mathcal{H}} - Z_T^*Z_T &= I - C_T^{1/2}T^*TC_T^{1/2} = C_T^{1/2}(C_T^{-1} - T^*T)C_T^{1/2} \\ &= C_T. \end{aligned}$$

Theorem 6.1. *Let $T \in \mathcal{L}(\mathcal{H})$ be a densely defined closed operator on \mathcal{H} . The operator Z_T has the following properties:*

(i) $Z_T \in \mathcal{B}(\mathcal{H})$, $\|Z_T\| \leq 1$ and

$$(6.3) \quad C_T = (I_{\mathcal{H}} + T^*T)^{-1} = I_{\mathcal{H}} - Z_T^*Z_T.$$

(ii) $(Z_T)^* = Z_T^*$.

(iii) If T is normal, then Z_T is normal.

Proof. The proof is based on the proof of Lemma 5.7 in [29] and is broken into steps.

Step 1: *Prove (i).*

First note that

$$(6.4) \quad \{C_T x : x \in \mathcal{H}\} = \mathcal{D}(I_{\mathcal{H}} + T^*T) = \mathcal{D}(T^*T).$$

Consequently, if $x \in \mathcal{H}$, then

$$\begin{aligned} \|TC_T^{1/2}C_T^{1/2}x\|^2 &= \langle T^*TC_Tx, C_Tx \rangle \\ &\leq \langle (I_{\mathcal{H}} + T^*T)C_Tx, C_Tx \rangle \\ &= \langle C_T^{-1}C_Tx, C_Tx \rangle \\ &= \langle x, C_Tx \rangle \\ &= \|C_T^{1/2}x\|^2. \end{aligned}$$

Thus, if $y \in \{C_T^{1/2}x : x \in \mathcal{H}\}$, then

$$(6.5) \quad \|Z_T y\| = \|TC_T^{1/2}y\| \leq \|y\|.$$

As $\text{Ker}(C_T) = \{0\}$, we have that $\text{Ker}(C_T^{1/2}) = \{0\}$ and thus $\{C_T^{1/2}x : x \in \mathcal{H}\}$ is a dense subset of \mathcal{H} . As T is a closed operator by assumption and $C_T^{1/2} \in \mathcal{B}(\mathcal{H})$, we get that Z_T is closed as well. Thus, we have $\{C_T^{1/2}x : x \in \mathcal{H}\} \subseteq \mathcal{D}(T)$, $\mathcal{D}(Z_T) = \mathcal{H}$ and, in view of (6.5), $\|Z_T\| \leq 1$.

Next, it follows from (6.5) and $C_T^{1/2}T^* \subseteq Z_T^*$ that

$$\begin{aligned} (I_{\mathcal{H}} - C_T)C_T^{1/2} &= C_T^{1/2}(I_{\mathcal{H}} + T^*T)C_T - C_T^{1/2}C_T \\ &= C_T^{1/2}T^*TC_T^{1/2}C_T^{1/2} \\ &\subseteq Z_T^*Z_TC_T^{1/2}. \end{aligned}$$

Thus, $Z_T^*Z_TC_T^{1/2} = (I_{\mathcal{H}} - C_T)C_T^{1/2}$ and, as $\{C_T^{1/2}x : x \in \mathcal{H}\}$ is a dense subset of \mathcal{H} , we get (6.3).

Step 2: *Prove (ii).*

Using (6.3) we get that $C_{T^*} = (I_{\mathcal{H}} + TT^*)^{-1}$. If $x \in \mathcal{D}(T^*)$, then let $y = C_{T^*}x$. Therefore,

$$x = (I_{\mathcal{H}} + TT^*)y$$

and

$$T^*x = T^*(I_{\mathcal{H}} + TT^*)y = (I_{\mathcal{H}} + T^*T)T^*y.$$

Thus, $C_{T^*}x \in \mathcal{D}(T^*)$ and hence

$$(6.6) \quad C_T T^*x = T^*y = T^*C_{T^*}x.$$

It follows easily from (6.6) and (6.3) that $p(C_{T^*})x \in \mathcal{D}(T^*)$ and

$$p(C_T)T^*x = T^*p(C_{T^*})x$$

for any real polynomial p of a real variable. By the Weierstrass approximation theorem, there exists a sequence of real polynomials $\{\phi_n\}_{n=0}^{\infty}$ which converge uniformly in supremum norm to the function $t \mapsto t^{1/2}$ on $[0, 1]$. Using Property (iv) of Theorem 3.7 we have that

$$\lim_{n \uparrow \infty} \|\phi_n(C_T) - C_T^{1/2}\| = \lim_{n \uparrow \infty} \|\phi_n(C_{T^*}) - C_{T^*}^{1/2}\| = 0.$$

Since T is a closed operator, T^* is also a closed operator. Thus, we have

$$\begin{aligned} C_T^{1/2}T^*x &= \lim_{n \uparrow \infty} \phi_n(C_T)T^*x = \lim_{n \uparrow \infty} T^*\phi_n(C_{T^*})x \\ &= T^*(C_{T^*})^{1/2}x \quad \text{for } x \in \mathcal{D}(T^*). \end{aligned}$$

As $C_T^{1/2}T^* \subseteq (TC_T^{1/2})^* = Z_T^*$, we get that

$$Z_{T^*}x = C_T^{1/2}T^*x = T^*(C_{T^*})^{1/2}x = (Z_T)^*x$$

for $x \in \mathcal{D}(T^*)$. Finally, since $\mathcal{D}(T^*)$ is dense in \mathcal{H} , we have that $Z_{T^*}x = (Z_T)^*x$, i.e., $Z_{T^*} = (Z_T)^*$.

Step 3: *Prove (iii).*

Using (6.2) on T and T^* and the fact that $TT^* = T^*T$ we have

$$I_{\mathcal{H}} - Z_T^* Z_T = (I_{\mathcal{H}} + T^* T)^{-1} = (I_{\mathcal{H}} + T T^*)^{-1} = I_{\mathcal{H}} - Z_{T^*}^* Z_{T^*}.$$

Making use of Property (ii) we have that

$$I_{\mathcal{H}} - Z_T^* Z_T = I_{\mathcal{H}} - Z_T Z_T^*,$$

i.e., Z_T is normal. \square

We are now ready to state and prove a spectral theorem for unbounded normal operators on a quaternionic Hilbert space.

Theorem 6.2. *Fix $j \in \mathbb{S}$ and let T be an unbounded right linear normal operator on \mathcal{H} , i.e., $T \in \mathcal{L}(\mathcal{H})$, $T \notin \mathcal{B}(\mathcal{H})$ and T is normal. There exists a Hilbert basis \mathcal{N}_j of \mathcal{H} and a unique spectral measure E so that if T is written with respect to \mathcal{N}_j , then*

$$(6.7) \quad T = \int_{\sigma_S(T) \cap \mathbb{C}_j^+} p dE(p).$$

Proof. The proof is broken into steps.

Step 1: *Show that a spectral measure E exists so that (6.7) holds.*

Let $\mathbb{B} = \{p \in \mathbb{H} : |p| < 1\}$, $\partial\mathbb{B} = \{p \in \mathbb{H} : |p| = 1\}$ and $\overline{\mathbb{B}} = \mathbb{B} \cup \partial\mathbb{B}$. If T is normal, then using Properties (i) and (iii) in Theorem 6.1 we get that $\|Z_T\| \leq 1$ and Z_T is normal, respectively. Thus, we may use Theorem 4.7 to obtain a Hilbert basis \mathcal{N}_j of \mathcal{H} so that $J = L_j$ and also a uniquely determined spectral measure F on $\sigma_S(Z_T) \cap \mathbb{C}_j^+$ so that

$$(6.8) \quad f(Z_T) = \int_{\sigma_S(Z_T) \cap \mathbb{C}_j^+} f(p) dF(p) \quad \text{for } f \in \mathcal{C}(\sigma_S(Z_T) \cap \mathbb{C}_j^+, \mathbb{C}_j).$$

In addition, it follows from Theorem 3.2.6 in [16] that

$$\sigma_S(Z_T) \subseteq \{p \in \mathbb{H} : |p| \leq \|Z_T\|\}$$

and hence

$$\sigma_S(Z_T) \cap \mathbb{C}_j^+ \subseteq \overline{\mathbb{B}} \cap \mathbb{C}_j^+.$$

If $x \in \mathcal{H}$ and $\sigma \in \mathfrak{B}(\sigma_S(T) \cap \mathbb{C}_j^+)$, then, in view of, item (v) in Lemma 5.3 and (6.8), we have

$$(6.9) \quad \langle (I_{\mathcal{H}} - Z_T^* Z_T) F(\sigma) x, F(\sigma) x \rangle = \int_{\sigma} (1 - |p|^2) d\langle F(p) x, x \rangle.$$

Recall that $I_{\mathcal{H}} - Z_T^* Z_T = (I_{\mathcal{H}} + T^* T)^{-1}$, i.e., $\text{Ker}(I_{\mathcal{H}} - Z_T^* Z_T) = \{0\}$. Thus, using (6.9) with

$$\sigma = \partial\overline{\mathbb{B}} \cap \mathbb{C}_j^+$$

we get that $\text{supp } F \subseteq \overline{\mathbb{B}} \cap \mathbb{C}_j^+$ and $F(\partial\mathbb{B} \cap \mathbb{C}_j^+) = 0$. Therefore,

$$F(\mathbb{B} \cap \mathbb{C}_j^+) = F[(\overline{\mathbb{B}} \cap \mathbb{C}_j^+) \setminus \partial(\mathbb{B} \cap \mathbb{C}_j^+)] = I_{\mathcal{H}}.$$

If $\varphi(p) = p(1 - |p|^2)^{-1/2}$, then $\varphi \in \mathcal{B}_{\infty}(F, \sigma_S(Z_T) \cap \mathbb{C}_j^+, \mathbb{C}_j)$. In view of item (iii) and (v) of Theorem 5.9 we have

$$\varphi(Z_T) = \left(\int_{\overline{\mathbb{B}} \cap \mathbb{C}_j^+} p dF(p) \right) \left(\int_{\overline{\mathbb{B}} \cap \mathbb{C}_j^+} \frac{1}{\sqrt{1 - |p|^2}} dF(p) \right)$$

and $\mathcal{D}(\varphi(Z_T)) = \mathcal{D}(g(Z_T))$, where $g \in \mathcal{B}_\infty(F, \sigma_S(Z_T) \cap \mathbb{C}_j^+, \mathbb{C}_j)$ is given by

$$g(p) = \frac{1}{\sqrt{1 - |p|^2}}.$$

Using Theorem 5.10, we have

$$g(T) = (1/g)(T)^{-1}.$$

Consequently, we may use item (i) in Corollary 4.8 to obtain

$$g(T) = \left\{ \left(\int_{\mathbb{B} \cap \mathbb{C}_j^+} (1 - |p|^2) dF(p) \right)^{1/2} \right\}^{-1}.$$

Putting these observations together, we obtain

$$(6.10) \quad \varphi(Z_T) = Z_T(C_T^{1/2})^{-1}.$$

Since $Z_T = TC_T^{1/2}$ we obtain $\varphi(Z_T) \subseteq T$. Using $C_T = (I_{\mathcal{H}} - Z_T^* Z_T)^{1/2}$, we get that $\varphi(Z_T) \subseteq T$. Thus, using Lemma 2.6 we get that

$$\varphi(Z_T) = T.$$

Let $E(\sigma) = F(\varphi^{-1}(\sigma))$, where

$$\varphi^{-1}(\sigma) = \{p \in \mathbb{H} : \varphi(p) \in \sigma\} \quad \text{for } \sigma \in \mathfrak{B}(\sigma_S(T) \cap \mathbb{C}_j^+).$$

It is readily checked that $E = F(\varphi^{-1})$ defines a spectral measure on \mathbb{C}_j^+ and thus using Lemma 5.11 we have

$$(6.11) \quad T = \int_{\mathbb{B} \cap \mathbb{C}_j^+} \varphi(p) dF(p) = \int_{\sigma_S(T) \cap \mathbb{C}_j^+} p dE(p).$$

Step 2: *Show that E from Step 1 is unique.*

If E and \tilde{E} are spectral measures on $\sigma_S(T) \cap \mathbb{C}_j^+$ which satisfy (6.7), then $F = E(\varphi)$ and $\tilde{F} = \tilde{E}(\varphi)$ are both spectral measures so that

$$(6.12) \quad Z_T = \int_{\mathbb{B} \cap \mathbb{C}_j^+} p dF(p) = \int_{\mathbb{B} \cap \mathbb{C}_j^+} p d\tilde{F}(p).$$

Let $\mathcal{P}(\sigma_S(T) \cap \mathbb{C}_j^+, \mathbb{C}_j)$ denote the space of \mathbb{C}_j -valued polynomials $\psi(p) = \sum_{a=0}^b \psi_a p^a$ on $\sigma_S(T) \cap \mathbb{C}_j^+$. In view of Lemma 5.3, (6.12) yields

$$\begin{aligned} \langle \psi(Z_T)x, x \rangle &= \int_{\sigma_S(T) \cap \mathbb{C}_j^+} \psi(p) d\langle F(p)x, x \rangle \\ &= \int_{\sigma_S(T) \cap \mathbb{C}_j^+} \psi(p) d\langle \tilde{F}(p)x, x \rangle, \quad \psi \in \mathcal{P}(\sigma_S(T) \cap \mathbb{C}_j^+, \mathbb{C}_j). \end{aligned}$$

As $\mathcal{P}(\sigma_S(T) \cap \mathbb{C}_j^+, \mathbb{C}_j)$ is a dense subset of $\mathcal{C}(\sigma_S(T) \cap \mathbb{C}_j^+, \mathbb{C}_j)$ we have that

$$\int_{\sigma_S(T) \cap \mathbb{C}_j^+} \psi(p) d\langle F(p)x, x \rangle = \int_{\sigma_S(T) \cap \mathbb{C}_j^+} \psi(p) d\langle \tilde{F}(p)x, x \rangle.$$

Finally, use the uniqueness in Theorem 4.7 to obtain that $F = \tilde{F}$ and hence $E = \tilde{E}$. \square

Corollary 6.3. *In the setting of Theorem 6.2, the following statements hold:*

- (i) *If $T \in \mathcal{L}(\mathcal{H})$ is a positive operator, then there exists a unique positive operator $W \in \mathcal{L}(\mathcal{H})$ so that $W^2 = T$.*
- (ii) *$T \in \mathcal{L}(\mathcal{H})$ is self-adjoint if and only if*

$$(6.13) \quad T = \int_{\mathbb{R}} t dE(t).$$

- (iii) *$T \in \mathcal{L}(\mathcal{H})$ is anti self-adjoint if and only if*

$$(6.14) \quad T = \int_{[0, \infty)} jt dE(t).$$

Proof. Using Theorem 6.2, the proof is completed as in Corollary 4.8. □

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